Pose Estimation in $\mathcal{G}(3,0,1)$
A Tutorial

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J. Russell Carpenter
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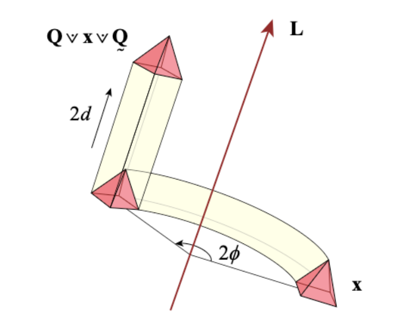
NASA Goddard Space Flight Center

Topics

Motors in $\mathcal{G}(3,0,1)$

Action of a Motor in $\mathcal{G}(3,0,1)$



(Image Credit: Eric Lengyel)

The motor $\mathbf{Q}$ rotates the object $\mathbf{x}$ about the (unitized) line $\mathbf{L}$ by the angle 2ϕ and translates it along the line by the distance $2d$.

Kinematics of a Motor in $\mathcal{G}(3,0,1)$

Exponential representation:

$$\mathbf{Q} = \mathrm{e}_{\veebar}^{(d+\phi\mathbb{1})\veebar\mathbf{L}}$$

Differentiating and letting $\mathbf{\Omega}$ denote the twist bivector, the kinematic equation for a motor becomes

$$\dot{\mathbf{Q}} = -\frac{1}{2}\mathbf{\Omega} \veebar \mathbf{Q}$$

If $\mathbf{L}$ is constant (over some time interval), then the twist bivector is

$$\begin{aligned} \mathbf{\Omega} &= -2(\dot{d} + \dot{\phi}\mathbb{1}) \veebar \mathbf{L} \\ &= -2(\mathbf{L}\dot{\phi} - \mathbf{L}^{\star}\dot{d}) \end{aligned}$$

where the superscript $\star$ indicates the weight dual.

Tensor Representation

Tensor Representation (Perwass)

- $\mathcal{G}(3,0,1)$ objects cast into order-1 tensors by stacking basis coefficients into $n \times 1$ column matrix
- $\mathcal{G}(3,0,1)$ operations occur via multiplication with higher-order tensors
- Example: geometric anti-product (order-3)

$$\mathbf{C} = \mathbf{A} \veebar \mathbf{B} \equiv c_k = \veebar_{kij} a_i b_j$$

- Contract on either i or j: Two ways to express above as a product of an order-2 and an order-1 tensor (i.e. a matrix-vector product):

$$c = \Psi_{\veebar}(a)b = \Xi_{\veebar}(b)a$$

- Similar relations hold for the wedge and and anti-wedge operations.
- Example: anti-reverse (order-2)

$$\mathbf{B} = \utilde{\mathbf{A}} \equiv b_j = \utilde{R}_{ji} a_i \equiv b = \utilde{R} a$$

- Example: Sandwich – does not reduce to a matrix-vector product

$$\mathbf{B} = \mathbf{Q} \veebar \mathbf{A} \veebar \utilde{\mathbf{Q}} \equiv b_\ell = \veebar_{kij} \veebar_{lkm} \utilde{R}_{mn} a_j q_i q_n$$
$$= M_{i\ell n} q_i q_n$$

Summary of Tensor Representations of $\mathcal{G}(3, 0, 1)$ Operations

Operation	Encoding Tensor	Matrix Representation
Geometric Product	$⟑_{kij}$	$\Psi_{⟑}(a) = ⟑_{kij} a_i$
		$\Xi_{⟑}(b) = ⟑_{kij} b_j$
Geometric Anti-Product	$⟇_{kij}$	$\Psi_{⟇}(a) = ⟇_{kij} a_i$
		$\Xi_{⟇}(b) = ⟇_{kij} b_j$
Wedge Product	$\bigwedge_{kij}$	$\Psi_{\wedge}(a) = \bigwedge_{kij} a_i$
		$\Xi_{\wedge}(b) = \bigwedge_{kij} b_j$
Anti-Wedge Product	$\bigvee_{kij}$	$\Psi_{\vee}(a) = \bigvee_{kij} a_i$
		$\Xi_{\vee}(b) = \bigvee_{kij} b_j$
Reverse	$\widetilde{R}_{ji}$	$\widetilde{R}$
Anti-Reverse	$\utilde{R}_{ji}$	$\utilde{R}$

“Point Solution” a.k.a. Solving the Generalized Wahba Problem

Generalized Wahba Problem

Given a set of multivector objects $\boldsymbol{M}^i$ and $\boldsymbol{N}^j$, corresponding to points, lines, planes, and/or motors, where $\boldsymbol{M}^i$ are members of a known model object set, and $\boldsymbol{N}^j$ are members of an observed object set, with known correspondences between $\boldsymbol{M}^i$ and $\boldsymbol{N}^j$, find the unique motor $\boldsymbol{Q}$ that represents the screw transformation of all the $\boldsymbol{M}^i$ into all the $\boldsymbol{N}^i$.

$$\boldsymbol{N}^i = \boldsymbol{Q} \mathbin{\dot{\vee}} \boldsymbol{M}^i \mathbin{\dot{\vee}} \underset{\sim}{\boldsymbol{Q}}$$

Point Solution (Perwass) i

Starting from

$$\boldsymbol{N}^i = \boldsymbol{Q} \mathbin{\dot\vee} \boldsymbol{M}^i \vee \underset{\sim}{\boldsymbol{Q}}$$

Multiply on the right by $\boldsymbol{Q}$ and subtract

$$\boldsymbol{N}^i \mathbin{\dot\vee} \boldsymbol{Q} - \boldsymbol{Q} \mathbin{\dot\vee} \boldsymbol{M}^i = \mathbf{0}$$

Cast this into linear algebra:

$$\Psi_{\dot\vee}(\boldsymbol{n}^i)\boldsymbol{q} - \Xi_{\dot\vee}(\boldsymbol{m}^i)\boldsymbol{q} = \mathbf{0}$$

Clear that $\boldsymbol{q}$ must be a null vector of $\Psi_{\dot\vee}(\boldsymbol{n}^i) - \Xi_{\dot\vee}(\boldsymbol{m}^i)$, which obviously holds for a sum over i.

Recognizing that $\boldsymbol{q}$ has only 8 non-zero values, let $\hat{\boldsymbol{q}} = H^\mathsf{T}\boldsymbol{q}$ select the non-zero elements.

The matrix

$$\sum_i \left(\Psi_{\triangledown}(\boldsymbol{n}^i) - \Xi_{\triangledown}(\boldsymbol{m}^i) \right) H$$

must have rank 7 for there to be a unique direction in the null space, but the length of this vector is not constrained.

For a proper motor, the constraint $\|\boldsymbol{q}_\circ\| = 1$ scales the entire null vector.

Let $\hat{\boldsymbol{q}}_\bullet = H_\bullet^\mathsf{T} \boldsymbol{q}_\bullet$ select the four non-zero elements of $\boldsymbol{q}_\bullet$ and let $\hat{\boldsymbol{q}}_\circ = H_\circ^\mathsf{T} \boldsymbol{q}_\circ$ select the four non-zero elements of $\boldsymbol{q}_\circ$.

The constraint that $\hat{\boldsymbol{q}}_\circ^\mathsf{T} H_\circ H_\bullet^\mathsf{T} \hat{\boldsymbol{q}}_\bullet = 0$ ensures that $\boldsymbol{q}_\bullet$ has the proper length.

Overdetermined Case

In either the overdetermined case, or the case of noisy observations, $\boldsymbol{q}$ will not be an exact null vector.

More general approach: seek the $\boldsymbol{q}$ that minimizes

$$J = \sum_i \beta_i \left\| \left(\Psi_\triangledown(\boldsymbol{n}^i) - \Xi_\triangledown(\boldsymbol{m}^i) \right) H \boldsymbol{q} \right\|^2$$

subject to the constraints $\|\boldsymbol{q}_\circ\| = 1$ and $\hat{\boldsymbol{q}}_\circ^\mathsf{T} H_\circ H_\bullet^\mathsf{T} \hat{\boldsymbol{q}}_\bullet = 0$.

Perwass shows that the maximum singular value of

$$\sum_i \left(\Psi_\triangledown(\boldsymbol{n}^i) - \Xi_\triangledown(\boldsymbol{m}^i) \right)^\mathsf{T} \left(\Psi_\triangledown(\boldsymbol{n}^i) - \Xi_\triangledown(\boldsymbol{m}^i) \right) H$$

provides the minimizing solution for $\boldsymbol{q}$.

Multiplicative Extended Kalman Filter

Small Perturbations

In terms of trig functions:

$$\mathbf{Q} = \mathbf{L}\sin\phi + \mathbb{1}\cos\phi - \mathbf{L}^{\star} d\cos\phi - d\sin\phi$$

Clear that for a motor $\delta\mathbf{Q}$ associated with a small rotation $\delta\phi$,

$$\begin{aligned} \delta\mathbf{Q} &\approx \mathbb{1} + \mathbf{L}\delta\phi - \mathbf{L}^{\star} d - d\delta\phi \\ &\approx \mathbb{1} + \delta\boldsymbol{\Theta} \end{aligned}$$

where $\delta\boldsymbol{\Theta} = \mathbf{L}\delta\phi - \mathbf{L}^{\star} d - d\delta\phi$.

If d is also small, then $d\delta\phi$ may be neglected as well.

Degrees of Freedom: $\mathbf{L}$ has only four degrees of freedom (two from its direction, one from the length of its moment, and one from the constraint that its moment and direction must be orthogonal), so $\delta\boldsymbol{\Theta}$ has only the additional two degrees of freedom from d and ϕ, for a total of six degrees of freedom.

Global and Local Estimates

Suppose there is an estimate of the motor, $\hat{\mathbf{Q}}$. True motor can be expressed as

$$\begin{aligned}\mathbf{Q} &= \delta\mathbf{Q} \veebar \hat{\mathbf{Q}} \\ &= (\mathbb{1} + \delta\boldsymbol{\Theta}) \veebar \hat{\mathbf{Q}}\end{aligned}$$

Take $\mathbf{Q}$ as the "global" pose representation, and $\delta\boldsymbol{\Theta}$ as the "local" representation of the pose errors (with translational error δd small enough to neglect $\delta d \delta\phi$).

The $\mathcal{G}(3, 0, 1)$ pose MEKF estimates $\delta\boldsymbol{\Theta}$, proceeding by the same three-step iteration as the original, attitude-only MEKF:

1. Measurement update of the local error, $\delta\boldsymbol{\Theta}$;
2. Reset that moves the updated information from the local error to the global pose estimate, $\hat{\mathbf{Q}}$; and
3. Time propagation step that moves the global pose estimate to the time of the next measurement(s).

Expressed in linear algebra, the measurement will in general be some nonlinear function of the true motor, corrupted by a measurement error, v:

$$y = h(q) + v$$

The measurement partials matrix is

$$\begin{aligned} H &= \frac{\partial h}{\partial q}\frac{\partial q}{\partial \delta\vartheta} \\ &= \frac{\partial h}{\partial q}\frac{\partial}{\partial \delta\vartheta}\left(\Psi_{\triangledown}\left(\begin{bmatrix} 0_{5\times 1} \\ \delta\vartheta \\ 0_{4\times 1} \\ 1 \end{bmatrix}\right)\hat{q}\right) = \frac{\partial h}{\partial q}\frac{\partial}{\partial \delta\vartheta}\left(\Xi_{\triangledown}(\hat{q})\begin{bmatrix} 0_{5\times 1} \\ \delta\vartheta \\ 0_{4\times 1} \\ 1 \end{bmatrix}\right) \\ &= \frac{\partial h}{\partial q}\Xi_{\triangledown}(\hat{q})\begin{bmatrix} 0_{5\times 6} \\ I_{6\times 6} \\ 0_{5\times 6} \end{bmatrix} = \frac{\partial h}{\partial q}\tilde{\Xi}_{\triangledown}(\hat{q}) \end{aligned}$$

Measurement Update ii

The measurement update for the error state is

$$\delta\vartheta^{+} = (I - KH)\delta\vartheta^{-} + K[y - h(\hat{q}^{-})]$$

where the gain K is the usual Kalman filter gain.

Covariance:

- Perwass develops a concept of a random multivector in terms of a linear algebra representation that is entirely consistent with the manner in which the traditional MEKF treats random quaternions.
- Covariance for the pose MEKF is therefore the 6×6 covariance of the error in $\delta\vartheta$.
- The usual Kalman filter relations for the covariance updates apply.

Measurement Partials for Direct Measurements

Suppose $\mathbf{Y}$ is a direct observation of a point, line or plane, and $\mathbf{X}$ is the representation of that same object in a model or reference system. Then the model object relates to the observed object according to

$$\mathbf{Y} = \mathbf{Q} \vee \mathbf{X} \vee \underset{\sim}{\mathbf{Q}} + \mathbf{V} \equiv y_\ell = M_{i\ell n} q_i q_n + v_\ell$$
$$y = h(q) + v$$

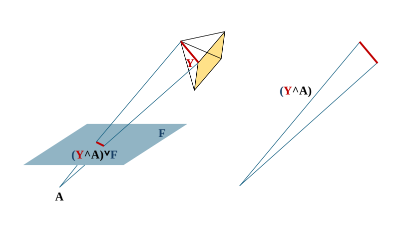
where $\mathbf{V} \equiv v$ is the observation error. The measurement partials matrix is

$$\frac{\partial h_\ell}{\partial q_p} = q_i \left(M_{ilp} + M_{pli} \right)$$
$$= L_{lp} + N_{pl}$$

so that the complete partial derivative with respect to the filter state is

$$H = \left(L_{lp} + N_{pl} \right) \tilde{\Xi}$$

Measurement Partials for Indirect Measurements i



An example of an indirect measurement is the observation of a point, line, or plane from an ideal pinhole camera. Let $\mathbf{A}$ denote the aperture point and $\mathbf{F}$ the virtual (front) image plane. The join of the observed object $\mathbf{Y}$ with the aperture point $\mathbf{Y} \wedge \mathbf{A}$, defines an object containing both objects. The intersection of this object with the virtual image plane is the observation:

$$
\begin{aligned}
\mathbf{Z} &= (\mathbf{Y} \wedge \mathbf{A}) \vee \mathbf{F} \\
&= \left[(\mathbf{Q} \mathbin{\dot\vee} \mathbf{X} \mathbin{\dot\vee} \underset{\sim}{\mathbf{Q}}) \wedge \mathbf{A} \right] \vee \mathbf{F} + \mathbf{V}
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
z_p &= q_i q_j M_{ikj} \bigwedge_{mk\ell} \bigvee_{pmn} a_\ell f_n + v_p \\
&= q_i q_j J_{ijp} + v_p \\
&= h_p(q) + v_p
\end{aligned}
$$

where $\bigwedge_{mk\ell}$ and $\bigvee_{pmn}$ encode the wedge and anti-wedge products, in a manner analagous to Γ_{kij}. Note that this relation does not enforce unitization on $\mathbf{Z} \equiv z$. The measurement partials matrix is

$$
\begin{aligned}
\frac{\partial h_p}{\partial q_r} &= q_i (J_{irp} + J_{rip}) \\
&= S_{rp} + T_{rp}
\end{aligned}
$$

Let $\mathbf{U_Z} \equiv u_z$ denote the unitized $\mathbf{Z} \equiv z$:

$$
\begin{aligned}
\mathbf{U_Z} = \mathbf{Z}/\sqrt{\mathbf{Z} \circ \mathbf{Z}} \equiv u_z &= z/\sqrt{z' \mathbb{G} z} \\
&= z/\|z\|_\circ
\end{aligned}
$$

The partial derivative of u_z with respect to z is

$$\frac{\partial u_z}{\partial z} = \frac{1}{\|z\|_\circ}\left(I - \frac{zz'\mathbb{G}}{\|z\|_\circ^2}\right)$$

so that the complete partial derivative of u_z with respect to the filter state is

$$H = \frac{1}{\|z\|_\circ}\left(I - \frac{zz'\mathbb{G}}{\|z\|_\circ^2}\right)(S_{rp} + T_{rp})\,\tilde{\Xi}$$

State Reset

The reset moves the update of $\delta\mathbf{\Theta}$ from the measurement to the global motor, according to

$$\hat{\mathbf{Q}}^{+} = (\mathbb{1} + \delta\mathbf{\Theta}) \veebar \hat{\mathbf{Q}}^{-}$$

The motor $(\mathbb{1} + \delta\mathbf{\Theta})$ must be unitized to ensure that $\hat{\mathbf{Q}}$ remains unitized across the reset. The usual MEKF practice is not to change the covariance when a reset occurs.

Differentiate $\mathbf{Q} = \delta\mathbf{Q} \vee \hat{\mathbf{Q}}$:

$$\dot{\mathbf{Q}} = \delta\dot{\mathbf{Q}} \vee \hat{\mathbf{Q}} + \delta\mathbf{Q} \vee \dot{\hat{\mathbf{Q}}}$$

Rearrange:

$$\delta\dot{\mathbf{Q}} = -\frac{1}{2}(\mathbf{\Omega} \vee \delta\mathbf{Q} - \delta\mathbf{Q} \vee \hat{\mathbf{\Omega}})$$

Approximate the true twist as $\mathbf{\Omega} \approx \hat{\mathbf{\Omega}} + \delta\mathbf{\Omega}$, where $\delta\mathbf{\Omega}$ has zero expectation; reduces to

$$\delta\dot{\mathbf{\Theta}} = \frac{1}{2}[\delta\mathbf{\Theta} \vee \hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}} \vee \delta\mathbf{\Theta}] - \frac{1}{2}\delta\mathbf{\Omega}\mathbb{1}$$

where the first term on the right-hand side is one of four possible commutators in $\mathcal{G}(3,0,1)$; these are sort of generalizations of the cross-product.

Finally, if $\delta\mathbf{\Theta}$ is zero at the beginning of a time update, as the reset imposes, it will remain zero, since $\delta\mathbf{\Omega}$ has zero expectation.

Pose MEKF Summary

1. Measurement update of the local error, $\delta\mathbf{\Theta}$:
    - Only need the 6 bivector components, $\hat{x} = \vartheta$
    - Usual Kalman Update cycles through any measurements available, beginning with $\hat{x}^{-} = 0$ due to reset:

$$
\begin{aligned}
K_k &= P_k^{-} H_k (H_k P_k^{-} H_k + V_k)^{-1} \\
\hat{x}_k^{+} &= \hat{x}_k^{-} + K_k (y_k - h(\hat{q}_k^{-})) \\
P_k^{+} &= (I - K_k H_k) P_k^{-} (I - K_k H_k)^{\mathsf{T}} + K_k V_k K_k^{\mathsf{T}}
\end{aligned}
$$

2. Reset:

$$
\hat{\mathbf{Q}}^{+} = (\mathbb{1} + \delta\mathbf{\Theta}) \veebar \hat{\mathbf{Q}}^{-}
$$

   then set $\hat{x} = \vartheta = 0$

3. Time propagation:
    - Advance global pose using $\dot{\mathbf{Q}} = -\frac{1}{2}\mathbf{\Omega} \veebar \mathbf{Q}$
    - Advance filter covariance: $\dot{P} = FP + PF^{\mathsf{T}} + Q$, where F is the bivector submatrix of $\frac{1}{2}[\Xi_{\veebar}(\hat{\Omega}) - \Psi_{\veebar}(\hat{\Omega})]$

Additional Details (Backups)

For the multi-vector $\boldsymbol{A}$,

$$\begin{aligned}\boldsymbol{A} = s\mathbf{e}_0 + p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4 \\ + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12} + v_z\mathbf{e}_{43} + v_y\mathbf{e}_{42} + v_x\mathbf{e}_{41} \\ + f_w\mathbf{e}_{321} + f_z\mathbf{e}_{412} + f_y\mathbf{e}_{431} + f_x\mathbf{e}_{423} + \sigma\mathbf{e}_{1234}\end{aligned}$$

express as order-1 tensor with the following coefficient ordering:

$$\begin{aligned}\boldsymbol{a} &= [s, p_x, p_y, p_z, p_w, m_x, m_y, m_z, v_z, v_y, v_x, f_w, f_z, f_y, f_x, \sigma]^\mathsf{T} \\ &= [s, \boldsymbol{p}_{xyz}^\mathsf{T}, p_w, \boldsymbol{m}_{xyz}^\mathsf{T}, \boldsymbol{v}_{zyx}^\mathsf{T}, f_w, \boldsymbol{f}_{zyx}^\mathsf{T}, \sigma]^\mathsf{T} \\ &= [s, \boldsymbol{p}_{xyzw}^\mathsf{T}, \boldsymbol{m}_{xyz}^\mathsf{T}, \boldsymbol{v}_{zyx}^\mathsf{T}, \boldsymbol{f}_{wzyx}^\mathsf{T}, \sigma]^\mathsf{T}\end{aligned}$$

Detailed Example: Matrix Representation of Geometric Anti-Product – 2 of 4

$$\Psi_{\dot\vee}(\boldsymbol{a}) = \begin{pmatrix}
\sigma & -f_x & -f_y & -f_z & -f_w & -v_x & -v_y & -v_z & -m_z & -m_y & -m_x & p_w & p_z & p_y & p_x & s \\
-f_x & \sigma & -v_z & v_y & m_x & -p_w & f_z & -f_y & p_y & -p_z & f_w & v_x & m_y & -m_z & s & p_x \\
-f_y & v_z & \sigma & -v_x & m_y & -f_z & -p_w & f_x & -p_x & f_w & p_z & v_y & -m_x & s & m_z & p_y \\
-f_z & -v_y & v_x & \sigma & m_z & f_y & -f_x & -p_w & f_w & p_x & -p_y & v_z & s & m_x & -m_y & p_z \\
0 & 0 & 0 & 0 & \sigma & 0 & 0 & 0 & -f_z & -f_y & -f_x & 0 & -v_z & -v_y & -v_x & p_w \\
v_x & p_w & -f_z & f_y & -p_x & \sigma & -v_z & v_y & m_y & -m_z & s & f_x & -p_y & p_z & -f_w & m_x \\
v_y & f_z & p_w & -f_x & -p_y & v_z & \sigma & -v_x & -m_x & s & m_z & f_y & p_x & -f_w & -p_z & m_y \\
v_z & -f_y & f_x & p_w & -p_z & -v_y & v_x & \sigma & s & m_x & -m_y & f_z & -f_w & -p_x & p_y & m_z \\
0 & 0 & 0 & 0 & -f_z & 0 & 0 & 0 & \sigma & v_x & -v_y & 0 & -p_w & -f_x & f_y & v_z \\
0 & 0 & 0 & 0 & -f_y & 0 & 0 & 0 & -v_x & \sigma & v_z & 0 & f_x & -p_w & -f_z & v_y \\
0 & 0 & 0 & 0 & -f_x & 0 & 0 & 0 & v_y & -v_z & \sigma & 0 & -f_y & f_z & -p_w & v_x \\
-p_w & -v_x & -v_y & -v_z & s & f_x & f_y & f_z & -p_z & -p_y & -p_x & \sigma & -m_z & -m_y & -m_x & f_w \\
0 & 0 & 0 & 0 & v_z & 0 & 0 & 0 & p_w & f_x & -f_y & 0 & \sigma & v_x & -v_y & f_z \\
0 & 0 & 0 & 0 & v_y & 0 & 0 & 0 & -f_x & p_w & f_z & 0 & -v_x & \sigma & v_z & f_y \\
0 & 0 & 0 & 0 & v_x & 0 & 0 & 0 & f_y & -f_z & p_w & 0 & v_y & -v_z & \sigma & f_x \\
0 & 0 & 0 & 0 & -p_w & 0 & 0 & 0 & -v_z & -v_y & -v_x & 0 & f_z & f_y & f_x & \sigma
\end{pmatrix}$$

Detailed Example: Matrix Representation of Geometric Anti-Product – 3 of 4

$$\Xi_{\triangledown}(\boldsymbol{a}) = \begin{pmatrix}
\sigma & f_x & f_y & f_z & f_w & -v_x & -v_y & -v_z & -m_z & -m_y & -m_x & -p_w & -p_z & -p_y & -p_x & s \\
f_x & \sigma & v_z & -v_y & -m_x & p_w & f_z & -f_y & -p_y & p_z & f_w & v_x & m_y & -m_z & -s & p_x \\
f_y & -v_z & \sigma & v_x & -m_y & -f_z & p_w & f_x & p_x & f_w & -p_z & v_y & -m_x & -s & m_z & p_y \\
f_z & v_y & -v_x & \sigma & -m_z & f_y & -f_x & p_w & f_w & -p_x & p_y & v_z & -s & m_x & -m_y & p_z \\
0 & 0 & 0 & 0 & \sigma & 0 & 0 & 0 & -f_z & -f_y & -f_x & 0 & -v_z & -v_y & -v_x & p_w \\
v_x & -p_w & -f_z & f_y & p_x & \sigma & v_z & -v_y & -m_y & m_z & s & -f_x & -p_y & p_z & f_w & m_x \\
v_y & f_z & -p_w & -f_x & p_y & -v_z & \sigma & v_x & m_x & s & -m_z & -f_y & p_x & f_w & -p_z & m_y \\
v_z & -f_y & f_x & -p_w & p_z & v_y & -v_x & \sigma & s & -m_x & m_y & -f_z & f_w & -p_x & p_y & m_z \\
0 & 0 & 0 & 0 & -f_z & 0 & 0 & 0 & \sigma & -v_x & v_y & 0 & -p_w & f_x & -f_y & v_z \\
0 & 0 & 0 & 0 & -f_y & 0 & 0 & 0 & v_x & \sigma & -v_z & 0 & -f_x & -p_w & f_z & v_y \\
0 & 0 & 0 & 0 & -f_x & 0 & 0 & 0 & -v_y & v_z & \sigma & 0 & f_y & -f_z & -p_w & v_x \\
p_w & -v_x & -v_y & -v_z & -s & -f_x & -f_y & -f_z & -p_z & -p_y & -p_x & \sigma & m_z & m_y & m_x & f_w \\
0 & 0 & 0 & 0 & v_z & 0 & 0 & 0 & p_w & -f_x & f_y & 0 & \sigma & -v_x & v_y & f_z \\
0 & 0 & 0 & 0 & v_y & 0 & 0 & 0 & f_x & p_w & -f_z & 0 & v_x & \sigma & -v_z & f_y \\
0 & 0 & 0 & 0 & v_x & 0 & 0 & 0 & -f_y & f_z & p_w & 0 & -v_y & v_z & \sigma & f_x \\
0 & 0 & 0 & 0 & -p_w & 0 & 0 & 0 & -v_z & -v_y & -v_x & 0 & f_z & f_y & f_x & \sigma
\end{pmatrix}$$

By denoting the usual skew-symmetric cross-product matrix as $K(\boldsymbol{x})$, and introducing the convention that reflecting the symbol for a matrix left-to-right indicates a similar reflection of its columns, $\Psi_{\vee}(\boldsymbol{a})$ and $\Xi_{\vee}(\boldsymbol{a})$ become

$$\Psi_{\vee}(\boldsymbol{a}) = \begin{pmatrix} \sigma & -\boldsymbol{f}_{xyz}^{\mathsf{T}} & -f_w & -\boldsymbol{v}_{xyz}^{\mathsf{T}} & -\boldsymbol{m}_{zyx}^{\mathsf{T}} & p_w & \boldsymbol{p}_{zyx}^{\mathsf{T}} & s \\ -\boldsymbol{f}_{xyz} & \sigma I + K(\boldsymbol{v}_{xyz}) & \boldsymbol{m}_{xyz} & -p_w I - K(\boldsymbol{f}_{xyz}) & f_w \reflectbox{I} + \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{v}_{xyz} & s\reflectbox{I} + \reflectbox{K}(\boldsymbol{m}_{xyz}) & \boldsymbol{p}_{xyz} \\ 0 & O_{1\times 3} & \sigma & O_{1\times 3} & -\boldsymbol{f}_{zyx}^{\mathsf{T}} & 0 & -\boldsymbol{v}_{zyx}^{\mathsf{T}} & p_w \\ \boldsymbol{v}_{xyz} & p_w I + K(\boldsymbol{f}_{xyz}) & -\boldsymbol{p}_{xyz} & \sigma I + K(\boldsymbol{v}_{xyz}) & s\reflectbox{I} + \reflectbox{K}(\boldsymbol{m}_{xyz}) & \boldsymbol{f}_{xyz} & -f_w \reflectbox{I} - \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{m}_{xyz} \\ O_{3\times 1} & O_{3\times 3} & -\boldsymbol{f}_{zyx} & O_{3\times 3} & \sigma I - K(\boldsymbol{v}_{zyx}) & O_{3\times 1} & -p_w I + K(\boldsymbol{f}_{zyx}) & \boldsymbol{v}_{zyx} \\ -p_w & -\boldsymbol{v}_{xyz}^{\mathsf{T}} & s & \boldsymbol{f}_{xyz}^{\mathsf{T}} & -\boldsymbol{p}_{zyx}^{\mathsf{T}} & \sigma & -\boldsymbol{m}_{zyx}^{\mathsf{T}} & f_w \\ O_{3\times 1} & O_{3\times 3} & \boldsymbol{v}_{zyx} & O_{3\times 3} & p_w I - K(\boldsymbol{f}_{zyx}) & O_{3\times 1} & \sigma I - K(\boldsymbol{v}_{zyx}) & \boldsymbol{f}_{zyx} \\ 0 & O_{1\times 3} & -p_w & O_{1\times 3} & -\boldsymbol{v}_{zyx}^{\mathsf{T}} & 0 & \boldsymbol{f}_{zyx}^{\mathsf{T}} & \sigma \end{pmatrix}$$

$$\Xi_{\vee}(\boldsymbol{a}) = \begin{pmatrix} \sigma & \boldsymbol{f}_{xyz}^{\mathsf{T}} & f_w & -\boldsymbol{v}_{xyz}^{\mathsf{T}} & -\boldsymbol{m}_{zyx}^{\mathsf{T}} & -p_w & -\boldsymbol{p}_{zyx}^{\mathsf{T}} & s \\ \boldsymbol{f}_{xyz} & \sigma I - K(\boldsymbol{v}_{xyz}) & -\boldsymbol{m}_{xyz} & p_w I - K(\boldsymbol{f}_{xyz}) & f_w \reflectbox{I} - \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{v}_{xyz} & -s\reflectbox{I} + \reflectbox{K}(\boldsymbol{m}_{xyz}) & \boldsymbol{p}_{xyz} \\ 0 & O_{1\times 3} & \sigma & O_{1\times 3} & -\boldsymbol{f}_{zyx}^{\mathsf{T}} & 0 & -\boldsymbol{v}_{zyx}^{\mathsf{T}} & p_w \\ \boldsymbol{v}_{xyz} & -p_w I + K(\boldsymbol{f}_{xyz}) & \boldsymbol{p}_{xyz} & \sigma I - K(\boldsymbol{v}_{xyz}) & s\reflectbox{I} - \reflectbox{K}(\boldsymbol{m}_{xyz}) & -\boldsymbol{f}_{xyz} & f_w \reflectbox{I} - \reflectbox{K}(\boldsymbol{p}_{xyz}) & \boldsymbol{m}_{xyz} \\ O_{3\times 1} & O_{3\times 3} & -\boldsymbol{f}_{zyx} & O_{3\times 3} & \sigma I + K(\boldsymbol{v}_{zyx}) & O_{3\times 1} & -p_w I - K(\boldsymbol{f}_{zyx}) & \boldsymbol{v}_{zyx} \\ p_w & -\boldsymbol{v}_{xyz}^{\mathsf{T}} & -s & -\boldsymbol{f}_{xyz}^{\mathsf{T}} & -\boldsymbol{p}_{zyx}^{\mathsf{T}} & \sigma & \boldsymbol{m}_{zyx}^{\mathsf{T}} & f_w \\ O_{3\times 1} & O_{3\times 3} & \boldsymbol{v}_{zyx} & O_{3\times 3} & p_w I + K(\boldsymbol{f}_{zyx}) & O_{3\times 1} & \sigma I + K(\boldsymbol{v}_{zyx}) & \boldsymbol{f}_{zyx} \\ 0 & O_{1\times 3} & -p_w & O_{1\times 3} & -\boldsymbol{v}_{zyx}^{\mathsf{T}} & 0 & \boldsymbol{f}_{zyx}^{\mathsf{T}} & \sigma \end{pmatrix}$$

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